ON THE ESSENTIAL LOGICAL STRUCTURE OF INTER-UNIVERSAL TEICHMÜLLER THEORY I, II, III, IV, V

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§1. <u>**Isogenies of elliptic curves and global multiplicative subspaces/canonical generators**</u>

(cf. [Alien], §2.3, §2.4; [ClsIUT], §1; [EssLgc], §3.2)

· A special case of Faltings' *isogeny invariance of the height* for *elliptic curves*

<u>Key assumption:</u>

$\exists$ **global multiplicative subspace** (GMS)

· *First key point of proof*:
(**invalid** for isogenies by **non-GMS** subspaces!!)

$$q \mapsto q^l \quad \text{(at primes of bad multiplicative reduction)}$$

$\ell \sim p$? . . . cf. **positive characteristic Frobenius morphism!**

. . . $\rightsquigarrow$ **"Gaussian" values** of **theta functions** in IUT

. . . $\rightsquigarrow$ need not only **GMS**, but also

. . . **global canonical generators (GCG)** (cf. §5)!

· *Second key point of proof*:

$$d\log(q) = \frac{dq}{q} \mapsto l \cdot d\log(q)$$

. . . yields **common** (cf. $\wedge$!) **container** (cf. **ampleness** of ω_E!)
for *both* elliptic curves!

. . . $\rightsquigarrow$ **log-link, anabelian geometry** in IUT

· One way to summarize IUT:

to generalize the above approach to **bounding heights**
via **theta functions** + **anabelian geometry**
to the case of *arbitrary elliptic curves*
by somehow **"simulating" GMS + GCG**!

§2. Gluings via Teichmüller dilations, inter-universality, and logical $\wedge/\vee$

(cf. [Alien], §2.11; [Alien], §3.3, (ii), (vi), (vii); [Alien], §3.11, (iv); [EssLgc], Examples 2.4.5, 2.4.7, 3.1.1; [EssLgc], §3.3, §3.4, §3.8 §3.11; [ClsIUT], §3)

· *Naive approach* to generalizing *Frobenius aspect* "$q^l \approx q$" of §1
— i.e., a situation in which, at the level of *arithmetic line bundles*, one may act as if there exists a *"Frobenius automorphism of the number field"* $q \mapsto q^l$ that *preserves arithmetic degrees*, while *at the same time multiplying them by* l *(!)*:

for $N \geq 2$ an integer, p a prime number, **glue** via "$*$" (cf. [Alien], §3.11, (iv); [EssLgc], Example 3.1.1; [EssLgc], §3.4):

$$^\dagger\mathbb{Z} \ni {}^\dagger p^N \leftarrow: * :\rightarrow {}^\ddagger p \in {}^\ddagger\mathbb{Z} \quad \ldots \text{ so } \quad (* \mapsto {}^\dagger p^N \in {}^\dagger\mathbb{Z}) \wedge (* \mapsto {}^\ddagger p \in {}^\ddagger\mathbb{Z})$$

... **not compatible** with **ring structures**!!

... but **compatible** with **multiplicative structures**,
actions of **Galois groups** as **abstract groups**!!

... **AND "$\wedge$"** depends on **distinct labels**!!

... ultimately, we want to **delete labels** (cf. §1!), but doing so *naively* yields — if one is to avoid giving rise to a **contradiction** "$p^N = p$"! — a *meaningless* **OR "$\vee$" indeterminacy**!!

$$(* \mapsto p^N \in \mathbb{Z}) \;\vee\; (* \mapsto p \in \mathbb{Z}) \qquad \Longleftrightarrow \qquad * \mapsto ?? \in \{p, p^N\} \subseteq \mathbb{Z}$$

(cf. *"contradiction"* asserted by
"redundant copies school (RCS)"!)

... in IUT, we would like to *delete the labels* in a somewhat more **"constructive" (!)** way!

· In IUT, we consider **gluing** via **Θ-link**, for l a prime number (cf. [Alien], §2.11; [Alien], §3.3, (ii), (vii); [EssLgc], §3.4, §3.8):

($\Theta^{\pm\mathrm{ell}}$NF-) Hodge theater, i.e., another model of conventional scheme theory surrounding given elliptic curve E	non-scheme- ——— theoretic Θ-link	($\Theta^{\pm\mathrm{ell}}$NF-) Hodge theater, i.e., model of conventional scheme theory surrounding given elliptic curve E

loc. unit gps.: $\quad G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}} \ \stackrel{\sim}{\to}\ G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$

loc. val. gps.: $\quad \left(\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}\right)^{\mathbb{N}} \ \stackrel{\sim}{\to}\ \left(\underline{\underline{q}}_{\underline{v}} \stackrel{\mathrm{def}}{=} q_{\underline{v}}^{\frac{1}{2l}}\right)^{\mathbb{N}}$

glob. val. gps.: corresponding *global realified Frobenioids* (s.t. product formula holds!)

... where $l \geq 5$ a prime number; $l^* \stackrel{\mathrm{def}}{=} \frac{l-1}{2}$;

E $(= E_F)$ is an elliptic curve over a number field F s.t. ... ;

$E[l] \subseteq E$ subgroup scheme of l-torsion points; $K \stackrel{\mathrm{def}}{=} F(E[l])$;

j_E is the *j-invariant* of E, so $F_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{Q}(j_E) \subseteq F$;

$\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$ collection of valuations of K s.t. ... ;

$q_{\underline{v}}$ denotes local q-parameter of E at bad (nonarch.) $\underline{v} \in \underline{\mathbb{V}}$;

$G_{\underline{v}}$ denotes the (local) absolute Galois group of $K_{\underline{v}}$ regarded **"inter-universally"** as an **abstract top. group**, i.e., **not** as a ("Galois"!) group of **field** automorphisms (cf. **incompatibility** with **ring structure**!);

$\mathcal{O}^{\times}_{\underline{\tilde{v}}}$: units of the ring of integers $\mathcal{O}_{\underline{\tilde{v}}}$ of an *algebraic closure* $K_{\underline{\tilde{v}}}$ of the completion $K_{\underline{v}}$ of K at $\underline{v}$;

$\mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\underline{\tilde{v}}}/\mathrm{tors}$ + "integral str." $\{\mathrm{Im}((\mathcal{O}^{\times}_{\underline{\tilde{v}}})^H)\}_{\mathrm{open}\ H \subseteq G_{\underline{v}}}$

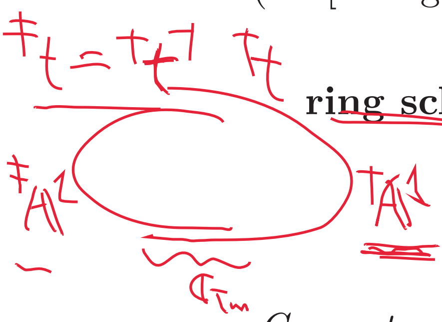
... note

two arithmetic/combinatorial dimensions of ring
= *one dilated dimension* + *another undilated dimension*

... cf. *cohomological dimension* of absolute Galois groups of *number fields* and *mixed characteristic local fields*, *topological dimension* of $\mathbb{C}^\times$!

$t \mp t^{-1}$

· *Concrete example of gluing*
(cf. [EssLgc], Example 2.4.7):



the **projective line** as a **gluing** of
ring schemes along a **multiplicative group scheme**

... cf. assertions of the **RCS**!

· *Concrete example of gluing*
(cf. [EssLgc], Example 3.3.1; [ClsIUT], §3; [Alien], §2.11):

classical complex Teichmüller deformations
of holomorphic structure
... cf. *two combinatorial/arithmetic dimensions of a ring*!!

... cf. assertions of the **RCS**!

· In IUT, we consider not just **Θ-link**, but also the **log-link**, which is defined, roughly speaking, by considering the

$\boldsymbol{p_{\underline{v}}}$**-adic logarithm** at each $\underline{v}$

(cf. [Alien], §3.3, (ii), (vi), Fig. 3.6; [EssLgc], §3.3, (InfH); [EssLgc], §3.11, (ΘORInd), (logORInd), (Di/NDi)), where we write $p_{\underline{v}}$ for the residue characteristic of (nonarch.) $\underline{v}$:

apply **same principle** as above of **label deletion** via
"**saturation** with **all possibilities** on either side of the link"

... but for Θ-*link*, this yields *meaningless* (ΘORInd)!!
... instead, consider "saturation" (logORInd) for log-*link*,
i.e., by constructing **invariants** for **log-link**
... where we recall that

log : **nondilated** unit groups $\rightleftarrows$ **dilated** value groups

... i.e., for *invariants*, "**nondilated** $\Longleftrightarrow$ **dilated**" ... cf. proof of §1!!

· The entire **log-theta-lattice** and the **"infinite H"** portion that is *actually used*:

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\dots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \dots \\
\dots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \dots \\
\dots \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, "\Theta"] & \dots \\
 & \vdots \arrow[u, "\mathfrak{log}"'] & \vdots \arrow[u, "\mathfrak{log}"'] &
\end{tikzcd}
\quad \supseteq \quad
\begin{tikzcd}
 & \vdots & \vdots & \\
 & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] & \\
(0,0) & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] \arrow[r, "\Theta", "\wedge"'] & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] & (1,0) \\
 & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] & \bullet \arrow[u, "\vee", "\mathfrak{log}"'] & \\
 & \vdots \arrow[u, "\vee", "\mathfrak{log}"'] & \vdots \arrow[u, "\vee", "\mathfrak{log}"'] &
\end{tikzcd}
$$

$$\left(\text{i.e., } \cancel{\textbf{not} \xrightarrow[\vee]{\Theta}}\ !\right)$$

... cf. Witt ring

$\bullet$ = {char p obj.}

$\uparrow \log$ = char p Fr

$\xrightarrow{\Theta}$ = $p^n/p^{n+1} \dots p^{n-1}/p^{n+1}$

§3. Symmetries/nonsymmetries and coricities of the log-theta-lattice

(cf. [Alien], §2.7, §2.8, §2.10, §3.2; [Alien], §3.3, (ii), (vi), (vii); [Alien], §3.6, (i); [EssLgc], §3.2, §3.3; [IUAni2])

· Fundamental Question:
So how do we construct **$\mathfrak{log}$-link invariants**?

· Fundamental Observations:
Θ-link (i.e., "$q^N \;\leftarrow:\; q$" for some $N \geq 2$) and
$\mathfrak{log}$-link (i.e., "*p-adic logarithm*" for some p)
clearly satisfy the following:

(1) **Θ-link**, **$\mathfrak{log}$-link** are **not compatible** with the **ring structures** in their *domains/codomains*;

(2) **Θ-link**, **$\mathfrak{log}$-link** are **not symmetric** with respect to **switching** their *domains/codomains*;

(3) **$\mathfrak{log}$-link $\circ$ Θ-link $\neq$ Θ-link $\circ$ $\mathfrak{log}$-link**;

(4) **$\mathfrak{log}$-link $\circ$ Θ-link $\neq$ Θ-link**

· **Frobenius-like** objects: objects whose definition **depends**, *a priori*, on the *coordinate* "$(n, m) \in \mathbb{Z} \times \mathbb{Z}$" of the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* at which they are defined (e.g., *rings*, *monoids*, etc. that do **not** map **isomorphically** via **Θ-link**, **$\mathfrak{log}$-link**)

· **Étale-like** objects: arise from *arithmetic (étale) fund. groups* regarded as *abstract topological gps.* ... cf. **inter-universality**!
$\Longrightarrow$ **mono-anabelian absolute anabelian geometry** may be applied (cf. *ampleness* of ω_E in §1!)
e.g.: inside each *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater* "$\bullet$", at each $\underline{v}$, $\exists$ a copy of the *arithmetic/tempered fundamental group*

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$$

of a certain finite étale covering of the *once-punctured elliptic curve* $X_{\underline{v}} \stackrel{\mathrm{def}}{=} E_{\underline{v}} \setminus \{\text{origin}\}$ (where $E_{\underline{v}} \stackrel{\mathrm{def}}{=} E \times_F K_{\underline{v}}$)

· **Étale-like** objects satisfy crucial **coricity** (i.e., "**common** — cf. $\wedge$! — to the domain/codomain")

  · each **$\mathfrak{log}$-link** induces **indeterminate** (cf. **inter-universality**!) isomorphisms

$$\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{v}}$$

  — cf. the evident *Galois-equivariance* of the (power series defining the) *p-adic logarithm*! — between copies in domain/codomain of the $\mathfrak{log}$-link

  · each **Θ-link** induces **indeterminate** (cf. **inter-universality**!) isomorphisms

$$G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}$$

  — i.e., "(Ind1)" — between copies in domain/codomain of the Θ-link

(so **abstract top. gps.** $\Pi_{\underline{v}}$, $G_{\underline{v}}$ are **coric** for **$\mathfrak{log}$-, Θ-links**!) and **symmetry** properties:

$$\begin{tikzcd}
\vdots & & \vdots \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[rr, "\Theta"] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\Pi_{\underline{v}} \arrow[r, two heads] & G_{\underline{v}} & \Pi_{\underline{v}} \arrow[l, two heads] \\
 & \circlearrowleft \ \mathrm{Aut}(G_{\underline{v}}) & \\
\vdots \arrow[uu, "\mathfrak{log}"'] & & \vdots \arrow[uu, "\mathfrak{log}"']
\end{tikzcd}$$

. . . **symmetric** w.r.t. dom./codom. of Θ-link!

· Thus, in summary, with regard to the desired **symmetry** and **coricity** properties:

Frobenius-like	**FALSE**	**FALSE**
étale-like	**TRUE**	**TRUE**

§4. Frobenius-like vs. étale-like structures and Kummer-detachment indeterminacies

(cf. [Alien], Examples 2.12.1, 2.12.3, 2.13.1; [Alien], §3.4; [Alien], §3.6, (ii), (iv); [Alien], §3.7, (i), (ii); [EssLgc], Examples 3.8.3, 3.8.4)

· **Kummer theory** yields *isoms.* between corresponding objects:

Frobenius-like objects $\stackrel{\sim}{\to}$ (mono-anabelian) étale-like objects

. . . but gives rise to **Kummer-detachment indeterminacies**, i.e., *one must pay some sort of price* for passing from

Frobenius-like objects that do *not* satisfy *coricity/symmetry* properties to *étale-like objects* that *do* satisfy *coricity/symmetry* properties

· In IUT, there are *three types* of *Kummer theory*:

(a) for **local units** $\mathcal{O}^{\times}_{\tilde{v}}$: classical Kummer theory via **local class field theory (LCFT)/Brauer groups** (cf. [Alien], Example 2.12.1);

(b) for **local theta values** $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}$: Kummer theory via **theta functions** and **Galois evaluation** at ***l*-torsion points** (cf. [Alien], §3.4, (iii), (iv));

(c) for **global field of moduli** F_{mod}: Kummer theory via **"κ-coric" algebraic rational functions** (essentially, non-linear polynomials w.r.t. some "point at infinity") and **Galois evaluation** at points defined over **number fields** (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))

· In general, *"Kummer theory"* proceeds by:

$$\left(\begin{array}{c}\text{extracting}\\ n\text{-th roots} \in M,\\ \text{for } n \in \mathbb{Z}_{>0}, \text{ of}\\ \textit{some element}\\ f \in \text{a } \textit{multipl.}\\ \textit{monoid } M\end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{c}\textit{Kummer class } \kappa_f\\ \in H^1\left(\left[\begin{array}{c}\text{some "Gal. group"}\\ \Pi \text{ that acts on } M\end{array}\right], \mu_n(M)\right)\end{array}\right)$$

. . . where $\mu_n(M)$ denotes n-torsion — i.e., *roots of unity*! — of M;

$\rightsquigarrow$ "$\widehat{\mathbb{Z}}$ version" by taking $\varprojlim_{n}$

· Main Substantive Issue: *eliminating* potential $\widehat{\mathbb{Z}}^\times$-**indeterminacy** from the conventional **cyclotomic rigidity isomorphism (CRI)**

“$\widehat{\mathbb{Z}}(1)$”

$$(\widehat{\mathbb{Z}} \cong) \quad \mu_{\widehat{\mathbb{Z}}}(M) \quad \xrightarrow{\sim} \quad \mu_{\widehat{\mathbb{Z}}}(\Pi) \quad (\cong \widehat{\mathbb{Z}})$$

arising from scheme theory (cf. [Alien], §3.4, (i), (ii), (iii), (iv))

... note that this is a *very substantive issue*! indeed,

indeterminate $\widehat{\mathbb{Z}}^\times$**-multiples/powers** of divs., line bdls., rational/merom. fns., elts. of number fields/local fields

completely destroy any notion of **positivity/inequalities** (recall that -1 lies in the closure of the natural numbers in $\widehat{\mathbb{Z}}$!) for **arithmetic degrees/heights**;

moreover, **inter-universality** — i.e., the property of “**not** *being* **anchored** *to*/**rigidified** *by any particular ring/scheme theory*” — means that the $\mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$ in the **Θ-link** (cf. §2) is subject to an *unavoidable* $\widehat{\mathbb{Z}}^\times$*-indeterminacy* “(Ind2)”

$$\widehat{\mathbb{Z}}^\times \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$$

... we shall refer to the **compatibility/incompatibility** — i.e., the **functorial equivariance/nonfunctoriality** — of a given Kummer theory with the *“inter-universality indeterminacies”* (Ind1), (Ind2) as the **multiradiality/uniradiality** of the Kummer theory; thus, the *multiradiality* of the Kummer theory may be understood as a sort of **“splitting/decoupling”** of the Kummer theory from the **unit group** $\mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$

· Another Substantive Issue for Cyclotomic Rigidity Isomorphisms: **compatibility** with the **profinite/tempered topology**, i.e., the property of admitting *finitely truncated versions*

$$(\mathbb{Z}/n\mathbb{Z} \cong) \quad \mu_n(M) \quad \xrightarrow{\sim} \quad \mu_n(\Pi) \quad (\cong \mathbb{Z}/n\mathbb{Z})$$

... this will be important since **ring structures** — which are necess. in order to define the *power series* for the *p-adic logarithm* (cf. **log-link**!) — only exist at *“finite n”* (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.3, 3.8.4), i.e.,

infinite “multiplicative Kummer towers $\varprojlim_n$*” destroy additive strs.*!

· In the case of the *three types* (a), (b), (c) of *Kummer theory* that are *actually used* in IUT (cf., especially, [Alien], Fig. 3.10; [Alien], §3.4, (v)):

(a) this approach to constructing CRI's is manifestly **compatible** with the **profinite topology**, but is **uniradial** since it depends in an essential way on the *extension of Galois modules* $1 \to \mathcal{O}^{\times}_{\underset{\sim}{\tilde{v}}} \to K^{\times}_{\underset{\sim}{\tilde{v}}} \to \mathbb{Q} \to 1$, hence is *fundamentally incompatible* with *indeterminacies* $\widehat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}^{\times}_{\underset{\sim}{\tilde{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\underset{\sim}{\tilde{v}}}$ (cf. [Alien], §3.4, (i));

(b) it follows from the theory of the **étale theta function** — in particular, the symmetries of **theta groups**, together with the **canonical splittings** arising from restriction to 2- (or, alternatively, 6-) torsion points — that this approach to constructing CRI's is both **compatible** with the **profinite/tempered topology** and **multiradial** (cf. [Alien], §3.4, (iii), (iv));

(c) it follows from elementary considerations concerning **"κ-coric" algebraic rational functions** that this approach to constructing CRI's is **multiradial**, but **incompatible** with the **profinite topology** (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))

· The *indeterminacies* $\widehat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}^{\times}_{\underset{\sim}{\tilde{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\underset{\sim}{\tilde{v}}}$ of (a) mean that the **theta values** and **elts.** $\in F_{\mathrm{mod}}$ obtained by **Galois evaluation**

$$\left(\boxed{\begin{array}{c}\text{Kummer class of some}\\ \text{sort of function}\end{array}}\right)\Bigg|_{\text{decomposition group of a point}}$$

in (b), (c) are *only meaningful* — i.e., *can only be protected* from the $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* — if they are considered, by applying the **"non-interference"** (up to roots of unity) of the monoids of (a) with those of (b) and (c), in terms of their actions on **log-shells**

(n, m)

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \stackrel{\mathrm{def}}{=} \frac{1}{2p_v}\log_{p_v}(\mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{v}}) \curvearrowleft F^{\times}_{\mathrm{mod}}$$

(n, m) (n, m)

... whose definition requires one to apply the $p_{\underline{v}}$*-adic logarithm*, i.e., the **log-link** *vertically shifted* by -1, relative to the coordin. "(n, m)" of the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater that gave rise to the *theta values* and *elements* $\in F_{\mathrm{mod}}$ under consideration (cf. [Alien], §3.7, (i)).

 (n, m)

· Here, we recall that only the **multiplicative monoid** $\mathcal{O}_{\underline{v}}^{\times\boldsymbol{\mu}}$ — i.e., *not* the *ring structures*, $\mathfrak{log}$-*link*, etc.! — is **accessible**, via the **common data** (cf. "$\wedge$!") in the gluing of the **Θ-link**, to the *opposite side* (i.e., domain/codomain) of the Θ-link!

Thus, to overcome the **vertical $\mathfrak{log}$-shift** discussed above, it is necessary to construct **invariants** w.r.t. the **$\mathfrak{log}$-link** (cf. §2!).

Here, we recall that **étale-like structures** "∘" — such as "$\Pi_{\underline{v}}$" — are indeed $\mathfrak{log}$-*link-invariant*, but the diagram — called the **$\mathfrak{log}$-Kummer correspondence** — arising from the *vertical column* (written *horizontally*, for convenience) in the *domain* of the Θ-*link*

$$\begin{tikzcd}
\ldots \arrow[r, "\mathfrak{log}"] & \bullet \arrow[r, "\mathfrak{log}"] \arrow[dr] & \bullet \arrow[r, "\mathfrak{log}"] \arrow[d] & \bullet \arrow[r, "\mathfrak{log}"] \arrow[dl] & \ldots \\
\ldots & & \circ & & \ldots
\end{tikzcd}$$

— where the vertical/diagonal arrows in the diagram are **Kummer isomorphisms** — is **not commutative**!

On the other hand, it is **upper semi-commutative** (!), i.e., all composites of **Kummer** and **$\mathfrak{log}$-link** morphisms on $\mathcal{O}_{\underline{v}}^{\times}$

$$\mathcal{O}_{\underline{v}}^{\times} \hookrightarrow \mathcal{O}_{\underline{v}} \hookrightarrow \mathcal{I}_{\underline{v}} \hookleftarrow \log_{p_{\underline{v}}}(\mathcal{O}_{\underline{v}}^{\times\boldsymbol{\mu}})$$

have images contained in the **log-shell** $\mathcal{I}_{\underline{v}}$ (cf. [Alien], Example 2.12.3, (iv)). This *very rough* variant of "commutativity" may be thought of as a type of **indeterminacy**, which is called "(Ind3)". It is (Ind3) that gives rise, ultimately, to the *upper bound* in the **height inequalities** that are obtained in IUT (cf. [Alien], Example 2.12.3, (iv); [Alien], §3.6, (iv); [Alien], §3.7, (i), (ii)).

· Thus, in summary, we have two **Kummer-detachment indeterminacies**, namely,

$$\text{(Ind2), (Ind3).}$$

§5. Conjugate synchronization and the structure of ($\Theta^{\pm\mathrm{ell}}$NF-) Hodge theaters

(cf. [Alien], §3.3, (ii), (iv), (v); [Alien], §3.4, (ii), (iii); [Alien], §3.6, (i), (ii), (iii); [AbsTopIII], §1; [EssLgc], §3.3; [EssLgc], Examples 3.3.2, 3.8.2, 3.8.3, 3.8.4; [ClsIUT], §3, §4; [IUTchI], Fig. I1.2)

- Fundamental Question:
  So **how** do we **"simulate"** **GMS** + **GCG**?

- In a word, we consider certain *finite étale coverings* over $K = F(E[l])$ of the *hyperbolic orbicurves*

$$X \stackrel{\text{def}}{=} E \setminus \{\text{origin}\}, \quad C \stackrel{\text{def}}{=} X//\{\pm 1\}$$

  determined by some *rank one quotient* $E[l]_K \twoheadrightarrow Q$:

$$\underline{X}_K \to X_K \stackrel{\text{def}}{=} X \times_F K \qquad \ldots \text{ determined by } E[l]_K \twoheadrightarrow Q$$

$$\underline{C}_K \to C_K \stackrel{\text{def}}{=} C \times_F K \qquad \ldots \text{ by taking } \underline{C}_K \stackrel{\text{def}}{=} \underline{X}_K//\{\pm 1\}$$

  ... where "//" denotes the *"stack-theoretic quotient"*

and restrict to **"local analytic sections"** of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ — called **"prime-strips"** (of which there are *various types*, as summarized in [IUTchI], Fig. I1.2), which may be thought of as a sort of *monoid-* or *Galois-theoretic* version of the classical notion of *adèles/idèles* — determined by various $\mathrm{Gal}(K/F)$-orbits of the *subset/section*

$$\mathbb{V}(K) \supseteq \underline{\mathbb{V}} \stackrel{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$$

where the *quotient* $E[l]_K \twoheadrightarrow Q$ is indeed the **"multipl. subspace"**, or where some *generator, up to* ± 1*, of* Q is indeed the **"canonical generator"**.

Working with such prime-strips means that many conventional objects associated to number fields — such as **absolute global Galois groups** or **prime decomposition trees** — much be *abandoned*! Indeed, this was precisely the *original motivation* (around 2005 - 2006) for the development of the **p-adic absolute mono-anabelian geometry** of [AbsTopIII], §1 [cf. [Alien], §3.3, (iv)]!

$$
\begin{array}{c|lcl|l}
\cline{2-4}
 & \cdot - \cdot - \cdot - \ldots - \cdot - \cdot - \cdot & = & \underline{\mathbb{V}} & \\
 & \cdot - \cdot - \cdot - \ldots - \cdot - \cdot - \cdot & \subseteq & \mathbb{V}(K) \setminus \underline{\mathbb{V}} & \curvearrowleft \ \mathrm{Gal}(K/F) \\
K & & \ldots & & \qquad \hookrightarrow \ GL_2(\mathbb{F}_l) \\
 & \cdot - \cdot - \cdot - \ldots - \cdot - \cdot - \cdot & \subseteq & \mathbb{V}(K) \setminus \underline{\mathbb{V}} & \\
 & \cdot - \cdot - \cdot - \ldots - \cdot - \cdot - \cdot & \subseteq & \mathbb{V}(K) \setminus \underline{\mathbb{V}} & \\
\cline{2-4}
\multicolumn{5}{c}{\downarrow} \\
\cline{2-4}
F_{\mathrm{mod}} & \cdot - \cdot - \cdot - \ldots - \cdot - \cdot - \cdot & = & \mathbb{V}(F_{\mathrm{mod}}) & \\
\cline{2-4}
\end{array}
$$

· The hyperbolic orbicurves $\underline{X}_K$, $\underline{C}_K$ admit **symmetries**

$$\mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\} \hookrightarrow \mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$$

... **additive/geometric!** (i.e., K-linear!)

$$\mathrm{Aut}(\underline{C}_K) \hookrightarrow \mathrm{Gal}(K/F) \twoheadrightarrow \mathbb{F}_l^{*} \stackrel{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}$$

... **multiplicative/arithmetic!**

obtained by considering the respective actions on cusps of $\underline{X}_K$, $\underline{C}_K$ that arise from elements of the *quotient* $E[l]_K \twoheadrightarrow Q$ [cf. [Alien], §3.3, (v); [Alien], §3.6, (i)]. At the level of *arithmetic fundamental groups*, these symmetries may be thought of as **finite groups** of **outer automorphisms** of

$$\Pi_{\underline{X}_K}, \quad \Pi_{\underline{C}_K}$$

— where we note that since, as is well-known, both the **geometric fundamental group** $\Delta_{\underline{X}_K}$ and the **global absolute Galois group** G_K are *slim* and do *not* admit *finite subgroups of order* > 2, these finite groups of outer automorphisms *do not lift to finite groups of (non-outer) automorphisms* (cf. [EssLgc], Example 3.8.2)!

Here, we note that since it is of *crucial importance* to **fix** the *quotient* $E[l]_K \twoheadrightarrow Q$ by the **"simulated GMS"**, we want to *start from* $\underline{C}_K$ and *descend*, via the *multiplic.* $\mathbb{F}_l^{\circledast}$*-symms.*, to $C_{F_{\mathrm{mod}}}$ (where $C_{F_{\mathrm{mod}}} \times_{F_{\mathrm{mod}}} F = C$), **not** the other way around, which would obligate us to consider **all Galois-**, hence, in particular, **all** $SL_2(\mathbb{F}_l)$**-conjugates** of Q. Note that this is precisely the **reverse** (!) order to proceed from the point of view of *classical Galois theory* (cf. [Alien], §3.6, (iii); [EssLgc], Ex. 3.8.2).

In particular, the *"strictly outer"* nature of the **multiplicative/arithmetic** $\mathbb{F}_l^{\circledast}$**-symmetries** means that various copies of the absolute local Galois groups "$G_{\underline{v}}$" (for, say, nonarch. $\underline{v} \in \underline{\mathbb{V}}$) in the prime-strips that are permuted by these symmetries can only be identified with one another **up to indeterminate inner automorphisms**, i.e., there is *no way to synchronize these conjugate indeterminacies* (cf. [Alien], §3.6, (iii); [EssLgc], Example 3.8.2).

By contrast, the "$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\bar{\underline{v}}}$" that appears in the *gluing data* for the Θ**-link** (cf. §2) must be **independent** of the "$j \in \mathbb{F}_l^{*}$" (cf. the "$\underline{q}^{j^2}$" of §2, where we think of this "j" as the smallest integer lifting $j \in \mathbb{F}_l^{*}$). That is to say, we need a **"conjugate synchronized"** $G_{\underline{v}}$ in order to construct the Θ**-link**, i.e., ultimately, in order to *express the LHS of the* Θ*-link in terms of the RHS*!! This is done by applying the **additive/geometric** $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.2, 3.8.3, 3.8.4).

Moreover, these *additive/geometric* $\mathbb{F}_l^{\rtimes\pm}$*-symms.* are **compatible**, rel. to the **log-link**, with the *crucial local CRI's/Galois eval.* of (a), (b) (but of (c) only up to **conj. indets.**! — cf. the $\mathbb{F}_l^{\circledast}$**-symm.** nature of (c) vs. the **non-**$\mathbb{F}_l^{\rtimes\pm}$**-symm.** nature of (b)!) of §4, *precisely* because these local CRI's of (a), (b) are *compatible* with the *profinite/tempered topology*, i.e., may be computed at a **finite truncated level**, where the **ring str.**, hence also the *power series* for the *p-adic logarithm*, is *well-defined* (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.3, 3.8.4).

Here, we recall that this *crucial property* of *compatibility with the profinite/tempered topology* in the case of (b), as opposed to (c), may be understood as a consequence of the fact that the **orders** of the **zeroes/poles at cusps** of the **theta function** are all equal to 1! Moreover, this phenomenon may in turn be understood as a consequence of the **symmetries** of **theta groups**, or, alternatively, as a consequence of the **quadratic form/first Chern class** "$\square^2$" in the exponent of the *classical series representation of the theta function* (cf. [Alien], §3.4, (iii), as well as the discussion below).

By contrast, in the case of (c), the orders of the zeroes/poles at cusps of the **algebraic rational functions** that are used differ from one another by arbitrary elements of $\mathbb{Z} \setminus \{0\}$ (cf. [Alien], §3.4, (ii))!

$$
\begin{array}{ccccc}
\begin{bmatrix} -l^* & < \ \dots \ < \ -1 \ < 0 \\ & < \ 1 \ < \ \dots \ < \ l^* \end{bmatrix} & & & & \begin{bmatrix} 1 \ < \ \dots \\ < \ l^* \end{bmatrix} \\
\Uparrow & & \Rightarrow \ \textbf{glue!} \ \Leftarrow & & \Uparrow \\
\underset{\curvearrowright}{\{\pm 1\}} \ \begin{pmatrix} -l^* & < \ \dots \ < \ -1 \ < 0 \\ & < \ 1 \ < \ \dots \ < \ l^* \end{pmatrix} & & & & \begin{pmatrix} 1 \ < \ \dots \\ < \ l^* \end{pmatrix} \\
\Downarrow & & & & \Downarrow \\
\begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \underset{\curvearrowright}{\mathbb{F}_l^{\rtimes \pm}} & \downarrow \\ \pm & \leftarrow & \pm \end{array} & & & & \begin{array}{ccc} * & \rightarrow & * \\ \uparrow & \underset{\curvearrowright}{\mathbb{F}_l^{*}} & \downarrow \\ * & \leftarrow & * \end{array}
\end{array}
$$

. . . **additive, geometric symmetries** . . . **multiplicative, arithmetic symmetries**

· The properties of **theta functions** in IUT discussed above are *particularly remarkable* when viewed from the point of view of the analogy with the **Jacobi identity** for the **theta function** on the *upper half-plane* (cf. [EssLgc], Example 3.3.2; [ClsIUT], §4). Indeed, on the one hand, the **quadratic form/first Chern class** "$\Box^2$" in the exponent of the *classical series representation of the theta function* (on the imaginary axis of the upper half-plane)

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

gives rise to the **theta group symmetries** that underlie the **rigidity properties** of theta functions that play a *central role* in IUT from the point of view of the ultimate goal in IUT of **expressing the LHS of the Θ-link in terms of the RHS** — i.e., *expressing the* **"Θ-pilot"** *on the LHS of the* Θ*-link* in terms *of the* **"q-pilot"** *on the RHS of the* Θ*-link.*

On the other hand, this **same quadratic form** in the exponent of the classical series representation of the theta function — which in fact appears as "$t \cdot \square^2$", i.e., with a factor t, where t denotes the standard coordinate on the imaginary axis of the upper half-plane — also underlies the well-known **Fourier transform invariance** of the **Gaussian distribution**, up to a sort of **"rescaling"**

$$t \cdot \square^2 \quad \mapsto \quad t^{-1} \cdot \square^2.$$

It is precisely this rescaling that gives rise to the *Jacobi identity*.

This state of affairs is *remarkable* (cf. [ClsIUT], §3, §4) in that the transformation $t \mapsto t^{-1}$ corresponds to the linear fractional transformation given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which, from the point of view of the analogy between the **"infinite H"** discussed at the end of §2 and the well-known *bijection*

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL_2^+(\mathbb{R})/\mathbb{C}^\times & \xrightarrow{\sim} & [0,1) \\ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) & \mapsto & \frac{\lambda-1}{\lambda+1} \end{array}$$

(where $\lambda \in \mathbb{R}_{\geq 1}$), may be understood as follows:

$$\begin{array}{cccc} \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) & \longleftrightarrow & \boldsymbol{\Theta}\textbf{-link} & \ldots \text{ cf. "}\textbf{not } \Theta\text{-link-invariants"!} \\ \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & \longleftrightarrow & \mathfrak{log}\textbf{-link} & \ldots \text{ cf. "}\mathfrak{log}\text{-link-invariants"!} \end{array}$$

(cf. [Alien], §3.3, (ii); [EssLgc], §3.3, (InfH), Example 3.3.2).

· <u>Concluding Question</u>:
So **why** do we need to **"simulate"** **GMS** + **GCG**?

... in order to secure the ***l*-torsion points** at which one conducts the **Galois evaluation** of the **étale theta function**, i.e., the *Kummer class* of the (reciprocal of the l-th root of the) *p-adic theta function* (cf. the discussion of the **Θ-link** in §2; §4, (b))

$$\underline{\underline{\Theta}}|_{l\text{-torsion points}} = \{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*}$$

... cf. the *classical series representation of the theta function* on the (imag. axis of the) upper half-plane — i.e., in essence, "$q = e^{2\pi i(it)}$"!

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t} = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}$$

§6. Multiradial representation and holomorphic hull

(cf. [Alien], §3.6, (iv), (v); [Alien], §3.7, (i), (ii); [EssLgc], §3.6, §3.10, §3.11; [ClsIUT], §2; [IUAni1])

· Fundamental Theme:
To *express/describe* the **Θ-pilot** on the LHS of the **Θ-link** *in terms of the RHS* of the Θ-link, while keeping the Θ-link itself **fixed** (!)

· For instance, the labels "j" in "$\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j=1,\dots,l^*}$" depend on the complicated **bookkeeping system** for these essen'ly **cuspidal labels** (i.e., labels of cuspidal inertia groups in the *geometric fundamental groups* $\Delta_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$) furnished (cf. §5) by the structure of the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater on the LHS*, which is **not accessible** from the point of view of the RHS. Thus, it is necessary to express these labels in a way that *is* accessible from the RHS, i.e., by means of **processions** of **capsules** of **prime-strips** "/"

$$/ \hookrightarrow // \hookrightarrow /// \hookrightarrow \quad \dots \quad \hookrightarrow \ / \dots /$$

(i.e., successive inclusions of *unordered* collections of prime-strips of incrementally increasing cardinality) — which still yield **symmetries** between the prime-strips at different labels without **"label-crushing"**, i.e., identifications between distinct labels (cf. [Alien], §3.6, (v)). We then consider the *actions* of (b), (c) (cf. §4) on **tensor-packets** of the *log-shells* arising from the data of (a) (cf. §4) inside each capsule:

$$\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j=1,\dots,l^*} \ \curvearrowright \ \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \ \curvearrowleft \ (F^{\times}_{\mathrm{mod}})_j$$

— where the *"tensor-packet"* is a tensor product of $j+1$ copies of $\mathcal{I}_{\underline{v}}$.

· In fact, the various monoids, Galois groups, etc. that appear in the data (a), (b), (c) of §4 — such as $\mathcal{I}_{\underline{v}}$, $\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j=1,\dots,l^*}$, $(F^{\times}_{\mathrm{mod}})_j$, etc. — come in **four types** (cf. [Alien], §3.6, (iv); [Alien], §3.7, (i)):

holomorphic Frobenius-like “(n,m)”: monoids etc. on which $\Pi_{\underline{v}} \curvearrowright$ acts, and whose construction involves the **ring structure** associated to the($\Theta^{\pm\text{ell}}$NF-)Hodge theater at $(n,m) \in \mathbb{Z} \times \mathbb{Z}$;

holomorphic étale-like “$(n,\circ)$”: similar data to (n,m), but reconstructed from $\Pi_{\underline{v}}$, hence **independent** of “m”;

mono-analytic Frobenius-like “$(n,m)^{\vdash}$”: monoids, etc., on which $G_{\underline{v}} \curvearrowright$ acts; used in the **gluing data** — called an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** — that appears in the **Θ-link**;

mono-analytic étale-like “$(n,\circ)^{\vdash}$”: similar data to $(n,m)^{\vdash}$, but reconstructed from $G_{\underline{v}}$, hence **independent** of “m” (and in fact also of “n”).

· Thus, in summary, the $\mathfrak{log}$-Kummer correspondence yields actions of the monoids of (b), (c) (cf. §4) on tensor-packets of log-shells arising from the data of (a) (cf. §4) up to the indeterminacy (Ind3)

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \quad \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \quad \curvearrowleft \quad (F^{\times}_{\text{mod}})_j$$

· *first*, at the level of objects of $(0,\circ)$; } descent

· then by **“descent”** (i.e., the observation that reconstructions from *certain input data* may in fact be conducted, up to natural isom., from *less/weaker input data*) up to indeterminacies (Ind1) at the level of objects of $(0,\circ)^{\vdash}$;

· then again by **“descent”** up to indeterminacies (Ind2) at the level of objects of $(0,0)^{\vdash} \xrightarrow{\sim} (1,0)^{\vdash}$ (via the Θ-link).

$$(0,0) \overset{(\text{Ind3})}{\rightsquigarrow} (0,\circ) \overset{(\text{Ind1})}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{(\text{Ind2})}{\rightsquigarrow} (0,0)^{\vdash} \xrightarrow[\sim]{\Theta\text{-link}} (1,0)^{\vdash}$$

(This last step involving (Ind2) plays the role of **fixing** the vertical coordinate, so that (Ind1), (Ind2) are **not mixed** with (Ind3) — cf. the discussion of “$\mathbb{C}^{\times}\backslash GL_2^{+}(\mathbb{R})/\mathbb{C}^{\times}$” at the end of §5!)

This is the **multiradial representation of the Θ-pilot** on the LHS of the Θ-link in terms of the RHS (cf. [Alien], §3.7, (i); [EssLgc], §3.10, §3.11). This multiradial representation plays the important role of **exhibiting** the (value group portion of the) **Θ-pilot** at $(0, 0)$ (i.e., which appears in the Θ-link!) as **one of the possibilities** within a **container** arising from the **RHS** of the **Θ-link** (cf. the *"infinite H"* at the end of §2; [EssLgc], §3.6, §3.10).

Next, by applying the operation of forming the **holomorphic hull** (i.e., *"$\mathcal{O}_{\underline{v}}$-module generated by"*) to the various *output regions* of the multiradial representation, we obtain a module over the local $\mathcal{O}_{\underline{v}}$'s on the RHS of the Θ-link. Then taking a suitable **root** of **"det(−)"** of this module yields an **arithmetic line bundle** — relative to the local $\mathcal{O}_{\underline{v}}$'s in the **zero label**! — in the *same category* as the category that gives rise to the ***q*-pilot** on the RHS of the Θ-link, *except* for a **vertical log-shift** by $+1$ in the 1-column (cf. the construction of *log-shells* from the "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\tilde{\underline{v}}}$'s" that appear in the *gluing data* of the Θ-link!) — cf. [EssLgc], §3.10.

Thus, by **symmetrizing** (i.e., with respect to vertical shifts in the 1-column) the procedure described thus far, we obtain a **closed loop**, i.e.,

$$\begin{array}{cccc}
 & \text{1-column} & \text{1-column} & \\
 & \vdots & \vdots & \\
 & \bullet & \bullet & \\
 & \uparrow \log & \uparrow \log & \\
 & \bullet & \bullet & \\
 & \uparrow \log & \uparrow \log & \\
 & \bullet & \bullet & (1,1) \\
 & \uparrow \log & \uparrow \log & \\
(1,0) & \bullet & \bullet & \\
 & \vdots & \vdots &
\end{array}$$

a situation in which the **distinct labels** on either side of the Θ-link (cf. the discussion at the beginning of §2!) may be **eliminated**, up to *suitable indeterminacies* (i.e., (Ind1), (Ind2), (Ind3); the holomorphic hull). In particular, by performing an entirely elementary **log-volume** computation, one obtains a **nontrivial height inequality**. This completes the proof of the *main theorems* of IUT (cf. [Alien], §3.7, (ii); [EssLgc], §3.10, §3.11).

Here, it is important to note that although the term "closed loop" at first might seem to suggest issues of **"diagram commutativity"** or **"log-volume compatibility"** — i.e., issues of

> *"How does one conclude a relationship between the* **output** *data and the* **input** *data of the* **closed loop***?"*

— in fact, such issues **simply do not exist** in this situation! That is to say, the *essential logical structure* of the situation

$$
\begin{array}{rcl}
A \wedge B & = & A \wedge (B_1 \vee B_2 \vee \ldots) \\
& \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \ldots \vee B'_1 \vee B'_2 \vee \ldots) \\
& \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \ldots \vee B'_1 \vee B'_2 \vee \ldots \vee B''_1 \vee B''_2 \vee \ldots) \\
& & \vdots
\end{array}
$$

proceeds by **fixing** the **logical AND "$\wedge$"** relation satisfied by the **Θ-link** and then adding various **logical OR "$\vee$" indeterminacies**, as illustrated in the following diagram (cf. [EssLgc], §3.10):

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & & (& \vee & \bullet & =) & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & (& \vee & \vee & \bullet & = & =) & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & (& \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=}) & = & = & \bullet & \\
\rightsquigarrow & & & (& \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & =) & = & \bullet & \\
\rightsquigarrow & & (& \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & =) & \bullet & \\
\rightsquigarrow & (& \vee & \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet &)
\end{array}
$$

§7. RCS-redundancy, Frobenius-like/étale-like strs., and Θ-/$\mathfrak{log}$-links

(cf. [Alien], §3.3, (ii); [EssLgc], Example 2.4.7; [EssLgc], §3.1, §3.2, §3.3, §3.4, §3.8, §3.11)

· **RCS ("redundant copies school") model of IUT** (i.e., "RCS-IUT" — cf. [EssLgc], §3.1):

This model ignores the various **crucial intertwinings of two dims.** in IUT (such as *addition/multiplication*, *local unit groups/value groups*, Θ-*link*/$\mathfrak{log}$-*link*, etc.).

Instead one works relative to a **single rigidified ring structure** by implementing, as described below, various **"RCS-identifications"** of **"RCS-redundant"** copies of objects — i.e., on the grounds that such RCS-identifications may be implemented *without affecting the essential logical structure of the theory* (cf. §2, §3!):

(RC-FrÉt) the **Frobenius-like** and **étale-like** versions of objects in IUT are **identified**;

(RC-$\mathfrak{log}$) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on opposite sides of the **$\mathfrak{log}$-link** in IUT are **identified**;

(RC-Θ) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on opposite sides of the **Θ-link** in IUT are **identified**.

Thus, locally, if

$\mathcal{O}_{\overline{k}}$ is the *ring of integers* of an *algebraic closure* $\overline{k}$ of $\mathbb{Q}_p$,
$k \subseteq \overline{k}$ is a *finite subextension* of $\mathbb{Q}_p$,
$\underline{\underline{q}} \in \mathcal{O}_k \stackrel{\mathrm{def}}{=} k \cap \mathcal{O}_{\overline{k}}$ is a *nonzero nonunit*,
$G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, and
Π ($\twoheadrightarrow G$) is the étale fundamental group of some *hyperbolic curve* (say, of strictly Belyi type) over k,

then we obtain the following situation:

<u>RCS-Θ-link</u>:

$$(k \supseteq) \quad (\underline{\underline{q}}^N)^{\mathbb{N}} \quad \stackrel{\sim}{\to} \quad \underline{\underline{q}}^{\mathbb{N}} \quad (\subseteq k)$$

... where the copies of "k", "$G \curvearrowright \overline{k}$", and "$G \curvearrowright \mathcal{O}_{\overline{k}}^{\times\mu}$" on opposite sides are **identified** (and in fact $N = 1^2, 2^2, \ldots, j^2, \ldots, (l^*)^2$, but we think of N as *some fixed integer* ≥ 2);

<u>RCS-$\mathfrak{log}$-link</u>:

$$(\overline{k} \supseteq) \quad \mathcal{O}_{\overline{k}}^{\times} \quad \stackrel{\log_p}{\longrightarrow} \quad \overline{k}$$

... where the copies of "k", "$\Pi \curvearrowright \overline{k}$", and "$\Pi \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$" on opposite sides are **identified**.

Then the *RCS-Θ-link* identifies

$$(0 \neq) \; N \cdot \mathrm{ord}(\underline{\underline{q}}) \;=\; \mathrm{ord}(\underline{\underline{q}}^N)$$

with $\mathrm{ord}(\underline{\underline{q}})$ (where $\mathrm{ord} : k^\times \to \mathbb{Z}$ is the valuation), which yields (since $N \neq 1$) a **"contradiction"**!

· <u>Elementary observation</u>: (cf. §2; [EssLgc], Example 3.1.1)
Let $^\dagger\mathbb{R}$, $^\ddagger\mathbb{R}$ be (*not necessarily distinct*!) copies of $\mathbb{R}$. Let $0 < x, y \in \mathbb{R}$; write $^\dagger x$, $^\ddagger x$, $^\dagger y$, $^\ddagger y$ for the corresponding elements of $^\dagger\mathbb{R}$, $^\ddagger\mathbb{R}$. If these two copies $^\dagger\mathbb{R}$, $^\ddagger\mathbb{R}$ of $\mathbb{R}$ are *distinct*, we may glue $^\dagger\mathbb{R}$ to $^\ddagger\mathbb{R}$ along

$$^\dagger\mathbb{R} \supseteq \{^\dagger x\} \stackrel{\sim}{\to} \{^\ddagger y\} \subseteq {}^\ddagger\mathbb{R}$$

without any *consequences* or *contradictions*. On the other hand, if $^\dagger\mathbb{R}$ and $^\ddagger\mathbb{R}$ are the *same copy of* $\mathbb{R}$, then to assert that $^\dagger\mathbb{R}$ is glued to $^\ddagger\mathbb{R}$ along

$$^\ddagger\mathbb{R} = {}^\dagger\mathbb{R} \supseteq \{^\dagger x\} \stackrel{\sim}{\to} \{^\ddagger y\} \subseteq {}^\ddagger\mathbb{R} = {}^\dagger\mathbb{R}$$

implies that we have a **contradiction**, unless $x = y$.

· Note that the **RCS-identification** (RC-Θ) discussed above may be regarded as analogous to identifying the two **distinct** copies of the **ring scheme** $\mathbb{A}^1$ that occur in the conventional gluing of these two distinct copies along the **group scheme** $\mathbb{G}_m$ to obtain $\mathbb{P}^1$. That is to say, the RCS-assertion of some sort of **logical equivalence**

$$\text{IUT} \quad \Longleftrightarrow \quad \text{RCS-IUT}$$

amounts to an assertion of an equivalence

$$\text{“}\mathbb{P}^1\text{”} \quad \Longleftrightarrow \quad \left(\begin{array}{c}\text{“}\mathbb{A}^1 \text{ regarded up to some sort of} \\ \text{identification of the standard coord.} \\ T \text{ with its inverse } T^{-1}\text{”}\end{array}\right)$$

(cf. §2; [EssLgc], Example 2.4.7) — i.e., which is *absurd*!

· **<u>Fundamental Problem with RCS-IUT</u>**:
(cf. [EssLgc], §3.2, §3.4, §3.8, §3.11)
There does **not exist** any **single “neutral” ring structure** with a single element “$*$” such that

$$(* = \underline{\underline{q}}^N) \quad \wedge \quad (* = \underline{\underline{q}})$$

Of course, there exists a *single “neutral” ring structure* with a single element “$*$” such that

$$(* = \underline{\underline{q}}^N) \quad \vee \quad (* = \underline{\underline{q}})$$

— but this requires one to contend, in RCS-IUT, with a fundamental (drastic!) **indeterminacy** (ΘORInd) that renders the entire theory (i.e., RCS-IUT, not IUT!) **meaningless**!

That is to say, the *essential logical structure* of IUT depends, in a very fundamental way, on the crucial **logical AND “$\wedge$”** property of the **Θ-link**, i.e., that the **abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip** in the Θ-link, regarded up to *isomorphism*, is *simultaneously* the **Θ-pilot** on the LHS of the Θ-link **AND** the **$\boldsymbol{q}$-pilot** on the RHS of the Θ-link.

This is possible precisely because the — *"weaker than ring" structures* given by — *realified Frobenioids* and *multiplic. monoids with abstract group actions* that constitute these Θ-/q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips are **isomorphic** — i.e., unlike the *"field plus distinguished element" pairs*

$$(k, \underline{\underline{q}}^N) \quad \text{and} \quad (k, \underline{\underline{q}}),$$

which are *not isomorphic*!

(... cf. the situation with $\mathbb{P}^1$: there does **not exist** a **single ring scheme** $\mathbb{A}^1$ with a single rational function "$*$" such that

$$(* = T^{-1}) \quad \wedge \quad (* = T).$$

There only exists a *single ring scheme* $\mathbb{A}^1$ with a single rational function "$*$" such that $(* = T^{-1}) \quad \vee \quad (* = T)$.)

Here, we note that the **RCS-identifications** of

G on opposite sides of the RCS-Θ-link or
Π on opposite sides of the RCS-$\mathfrak{log}$-link

— which arise from **Galois-equivariance** properties with respect to the **single "neutral" ring structure** discussed above, i.e., which is subject to the (drastic!) **(ΘORInd) indeterminacies** — yield **false symmetry/coricity** (such as the symmetry of "$\Pi \twoheadrightarrow G \twoheadleftarrow \Pi$") properties, i.e., *false* versions of the symm./cor. props. discussed in §3.

Indeed, the various **Galois-rigidifications** — i.e., embeddings of the abstract topological groups involved into the group of automorphisms of **some field** — that *underlie these Galois-equivariance or false symmetry/coricity properties* are **unrelated** to the Galois-rigidifications that underlie the ("true"!) corresponding symmetry/coricity properties of §3. That is to say, setting up a situation in which these ("true"!) symm./cor. props. of §3 do indeed hold is the whole point of the notion of **"inter-universality"**, i.e., working with *abstract groups*, *abstract monoids*, etc.!

· Finally, we observe that (cf. [Alien], §3.3, (ii); [EssLgc], §3.3)

the **very definition** of the **$\mathfrak{log}$-link**, **Θ-link** (cf. §2;
$\mathfrak{log}$: **nondilated** unit groups $\rightleftarrows$ **dilated** value groups!)

$\Longrightarrow$ the **falsity** of (RC-$\mathfrak{log}$):

Indeed, there is **no natural way** to relate the *two* Θ*-links* (i.e., the *two horizontal arrows* below) that emanate from the *domain* and *codomain* of the $\mathfrak{log}$*-link* (i.e., the *left-hand vertical arrow*)

$$\begin{tikzcd} \bullet \arrow[r, "\Theta"] & \bullet \arrow[d, dash, dotted, "??"] \\ \bullet \arrow[u, "\mathfrak{log}"] \arrow[r, "\Theta"] & \bullet \end{tikzcd}$$

— that is to say, there is *no natural candidate* for "??" (i.e., such as, for instance, an *isomorphism* or the $\mathfrak{log}$*-link* between the two bullets "$\bullet$" on the *right-hand side* of the diagram) that makes the diagram *commute.* Indeed, it is an easy exercise to show that *neither* of these candidates for "??" yields a commutative diagram.

- Analogy with classical complex Teichmüller theory:
  (cf. [EssLgc], Example 3.3.1)

Let $\lambda \in \mathbb{R}_{>1}$. Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory

$$\begin{array}{ccccccc} & \Lambda : \mathbb{C} & \to & \mathbb{C} & & & \\ \mathbb{C} & \ni \ z = x + iy & \mapsto & \zeta \ = \ \xi + i\eta & \stackrel{\mathrm{def}}{=} & \lambda \cdot x + iy \ \in & \mathbb{C} \end{array}$$

— where $x, y \in \mathbb{R}$. Let $n \geq 2$ be an integer, ω a *primitive* n*-th root of unity.* Write $(\omega \in)\ \mu_n \subseteq \mathbb{C}$ for the group of n-th roots of unity. Then *observe* that

> if $n \geq 3$, then there does *not* exist $\omega' \in \mu_n$ such that $\Lambda(\omega \cdot z) = \omega' \cdot \Lambda(z)$ for all $z \in \mathbb{C}$.

(Indeed, this *observation* follows immediately from the fact that if $n \geq 3$, then $\omega \notin \mathbb{R}$.) That is to say, in words,

> Λ is **not compatible** with multiplication by μ_n unless $n = 2$ (in which case $\omega = -1$).

This *incompatibility* with **"indeterminacies"** arising from multiplication by μ_n, for $n \geq 3$, may be understood as one fundamental reason for the *special role* played by **square differentials** (i.e., as opposed to n-th power differentials, for $n \geq 3$) in classical complex Teichmüller theory.

§8. Chains of gluings/logical $\wedge$ relations
(cf. [EssLgc], §3.5, §3.6, §3.11; [ClsIUT], §2)

· Fundamental Question:
Why is the **logical AND "$\wedge$"** relation of the Θ-link so *fundamental* in IUT?

· Consider, for instance, the *classical theory of* **crystals** (cf. [ClsIUT], §2; [EssLgc], §3.5, (CrAND), (CrOR), (CrRCS)):

The *"crystals"* that appear in the conventional theory of crystals may be thought of as **"$\wedge$-crystals"**. Alternatively, one could consider the (in fact *meaningless*!) theory of **"$\vee$-crystals"**. One verifies easily that this theory of *"$\vee$-crystals"* is in fact essentially equivalent to the theory obtained by replacing the various **thickenings of diagonals** that appear in the conventional theory of crystals by the "$(-)_{\mathrm{red}}$" of these thickenings, i.e., by the **diagonals themselves**! Finally, we observe that consideration of *"$\vee$-crystals"* corresponds to the **indeterminacy** (ΘORInd) that appears in RCS-IUT, i.e.:

$$\begin{array}{ccc} \textbf{IUT} & \longleftrightarrow & \text{“}\wedge\textbf{-crystals”} \\ \textbf{RCS-IUT} & \longleftrightarrow & \text{“}\vee\textbf{-crystals”} \end{array}$$

· Frequently Asked Question:
In IUT, one starts with the fundamental **logical AND "$\wedge$"** relation of the Θ-link, which holds precisely because of the **distinct labels** on the *domain/codomain* of the Θ-link. Then what is the the **minimal** amount of **indeterminacy** that one must introduce in order to **delete** the **distinct labels** without invalidating the fundamental *logical AND "$\wedge$"* relation?

In short, the answer (cf. §6!) is that one needs **(Ind1), (Ind2), (Ind3),** together with the operation of forming the **holomorphic hull**. In some sense, the most fundamental of these indets. is

(Ind3),

which in fact in some sense **"subsumes"** the other indeterminacies — at least **"to highest order"**, i.e., in the *height inequalities* that are ultimately obtained (cf. [EssLgc], §3.5, (CnfInd1+2), (CnfInd3); [EssLgc], §3.11, (Ind3>1+2)).

Recall from §4 that (Ind3) is an inevitable consequence of the **non-commutativity** of the **$\mathfrak{log}$-Kummer correspondence**

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

(cf. also the discussion of the *falsity* of (RC-$\mathfrak{log}$), (RC-FrÉt) in §7!). On the other hand, observe that since automorphisms of the (topological module constituted by the) **log-shell** $\mathcal{I}_{\underline{v}}$ *always preserve* the submodule

$$p^n \cdot \mathcal{I}_{\underline{v}}$$

(where $n \geq 0$ is an integer) — i.e., even if they do *not* necessarily preserve $\mathcal{O}_{\underline{v}} \subseteq \mathcal{I}_{\underline{v}}$ or positive powers of the *maximal ideal* $\mathfrak{m}_{\underline{v}} \subseteq \mathcal{O}_{\underline{v}}$! — it follows immediately that

(Ind1) (or, *a fortiori*, the "$\Pi_{\underline{v}}$ version" of (Ind1) — cf. the discussion of (Ind1) in §3) and

(Ind2)

(both of which induce automorphisms of $\mathcal{I}_{\underline{v}}$) can **never account for** any sort of **"confusion"** (cf. the definition of the Θ-*link*) between

$$\text{“}\underline{\underline{q}}_{\underline{v}}^{(l^*)^2}\text{”} \text{ and } \text{“}\underline{\underline{q}}_{\underline{v}}\text{”}$$

(cf. [EssLgc], §3.5, (CnfInd1+2), (CnfInd3); [EssLgc], Example 3.5.1; [EssLgc], §3.11, (Ind3>1+2))! This is a *common misunderstanding*!

· Now let us return to the *Fundamental Question* posed above.

We begin our discussion by observing (cf. [EssLgc], §3.6) that

($\wedge$-Chn) the logical structure of IUT proceeds by *observing a* **chain** *of* **AND relations** *"$\wedge$"* (*not* a chain of *intermediate inequalities*! — cf. [EssLgc], §3.6, (Syp3)).

That is to say, one starts with the **logical AND "$\wedge$"** relation of the Θ-link. This *logical AND "$\wedge$"* relation is *preserved* when one passes to the **multiradial representation of the Θ-pilot** as a consequence of the following fact:

($\wedge$-Input) the **input data** for this multiradial algorithm consists solely of an **abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip**; moreover, this multiradial algorithm is **functorial** with respect to arbitrary isomorphisms between $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips.

Indeed, at a more technical level, we make the *fundamental observation* that this multiradial algorithm proceeds by *successive application*, in one form or another, of the following principle of **"extension of indeterminacies"**:

(ExtInd) If A, B, and C are propositions, then it holds (that $B \implies B \vee C$ and hence) that

$$A \wedge B \implies A \wedge (B \vee C).$$

(cf. the final portion of §6!). Applications of (ExtInd) may be further *subclassified* into the following *two types*:

(ExtInd1) (*"set-theoretic"*) operations that consist of simply adding **more possibilites/indeterminacies** (which corresponds to passing from B to $B \vee C$) within some **fixed container**;

(ExtInd2) (*"stack-theoretic"*) operations in which one **identifies** (i.e., *"crushes together"*, by passing from B to $B \vee C$) objects with **distinct labels**, at the cost of passing to a situation in which the object is regarded as being only known **up to isomorphism**

(cf. the discussion of §9 below).

At this point, we recall from §6 that the *ultimate goal* of various applitions of (ExtInd) in the algorithms that constitute the **multiradial representation of the Θ-pilot** is to

exhibit the (value group portion of the) **Θ-pilot** at $(0, 0)$ (i.e., which appears in the Θ-link!) as **one of the possibilities** within a **container** arising from the **RHS** of the **Θ-link**

(cf. the situation surrounding *rational functions* on $\mathbb{P}^1$, as discussed in [EssLgc], Example 2.4.7, (ii)!).

In particular, any problems in understanding the *essential logical str.* of IUT (i.e., the argument of §6) may be *diagnosed/analyzed* by asking the following **diagnostic question**:

($\wedge$-Dgns) **precisely where** in the finite sequence of steps that appear is the **first step** at which the person feels that the **preservation** of the **crucial AND relator** "$\wedge$" is *no longer clear*?

§9. Poly-morphisms, descent to underlying strs., and inter-universality
(cf. [EssLgc], Example 3.1.1; §3.7, §3.8, §3.9, §3.11)

· In IUT, one often considers **poly-morphisms**, i.e., sets of morphisms between objects — such as **full poly-isomorphisms** (the set of all isomorphisms between two objects) — as a tool to keep track explicitly of **all possibilities** that appear. Classical examples include **homotopy classes** of continuous maps in topology and **outer homomorphisms** (i.e., homomorphisms considered up to composition with inner automorphisms). Roughly speaking, working with *full poly-isomorphisms* corresponds to *"considering objects up to isomorphism"*. From the point of view of the *chains of* $\wedge$*'s/* $\vee$*'s*

$$\begin{aligned} A \wedge B \quad &= \quad A \wedge (B_1 \vee B_2 \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots \vee B_1'' \vee B_2'' \vee \dots) \\ &\qquad \vdots \end{aligned}$$

discussed in §6, consideration of poly-morphisms corresponds to adding to the *collection of possibilities*, i.e., to the *collection of* $\vee$*'s* that appear (cf. *"set-theoretic" (ExtInd1)*!) — cf. [EssLgc], §3.7.

· One fundamental aspect of IUT lies in the use of numerous **functorial algorithms** that consist of the construction

$$\textit{input data} \quad \rightsquigarrow \quad \textit{output data}$$

of certain *output data* associated to given *input data*. Often it is natural to regard the *"input data"* as *"original data"* and to regard the *"output data"* as *"underlying data"*:

$$\begin{array}{ccc} \textit{input data} & \rightsquigarrow & \textit{output data} \\ \| & & \| \\ \textit{original data} & & \textit{underlying data} \end{array}$$

One important example of this sort of situation in IUT involves the notion of **"q-/Θ-intertwinings"** on an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* (cf. [EssLgc], §3.9):

original data (“equipped with an intertwining”):
the **q-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip (in the case of the “q-intertwining”) or the **Θ-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip (in the case of the “Θ-intertwining”), equipped with the *auxiliary data* of how this q-/Θ-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip is constructed from some *($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater*;

underlying data:
the *abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip* associated to the above *original data*, i.e., obtained by *forgetting* the *auxiliary data*.

· In general, in any sort of situation involving *original/underlying data*, it is natural to consider the issue of **descent** to (a functorial algorithm in) the *underlying data* of a **functorial algorithm** in the *original data*: we say that

a *functorial algorithm* Φ in the *original data* **descends** to a *functorial algorithm* Ψ in the *underlying data* if there exists a functorial isomorphism

$$\Phi \quad \overset{\sim}{\to} \quad \Psi|_{original\ data}$$

between Φ and the *restriction* of Ψ, i.e., relative to the given construction *original data* $\quad\rightsquigarrow\quad$ *underlying data*.

That is to say, roughly speaking, to say that the functorial algorithm Φ in the original data *descends* to the *underlying data* means, in essence, that although the construction constituted by Φ depends, *a priori*, on the **“finer”** *original data*, in fact, up to *natural isomorphism* (cf. *“stack-theoretic” (ExtInd2)*!), the functorial algorithm only depends on **“coarser”** *underlying data*.

· One elementary example of *descent* is the following (cf. [EssLgc], Example 3.9.1):

Let X be a *scheme*, T a *topological space*. Write

· $|X|$ for the *underlying topological space* of X,

· $\mathrm{Open}(X)$ for the category of *open subschemes* of X and *open immersions* over X,

· $\mathrm{Open}(T)$ for the category of *open subsets* of T and *open immersions* over T.

Then the *functorial algorithm*

$$X \quad \mapsto \quad \mathrm{Open}(X)$$

— defined, say, on the category of schemes and morphisms of schemes — *descends*, relative to the construction $X \rightsquigarrow |X|$, to the *functorial algorithm*

$$T \quad \mapsto \quad \mathrm{Open}(T)$$

— defined, say, on the category of topological spaces and continuous maps of topological spaces. That is to say, there is a *natural functorial isomorphism*

$$\mathrm{Open}(X) \quad \stackrel{\sim}{\rightarrow} \quad \mathrm{Open}(|X|)$$

(i.e., more precisely, following the conventions employed in IUT, a *natural functorial isomorphism class of equivalences of categories*) — cf. *(ExtInd2)*!

· **Inter-universality** in IUT — cf. the *abstract topological groups/monoids* (as opposed to *Galois groups/multiplicative monoids of rings*!) that appear in the Θ-link, as discussed in §2, §3, §4, §7 — arises from the fact that the structures **common** (cf. "$\wedge$"!) to both sides of the **Θ-link** are **weaker** than ring structures. On the other hand, despite this *"ring str. vs. weaker than ring str."* difference, at a *purely foundational level*, the resulting indeterminacies (i.e., (Ind1), (Ind2)) are in fact *completely qualitatively similar* to the **inner automorphism indeterminacies** in [SGA1] (cf. [EssLgc], §3.8).

In this context, it is useful to recall the elementary fact that these inner automorphism indeterminacies are *unavoidable* (cf. [EssLgc], Example 3.8.1, (i)!):

Let

k be a *perfect field*;

$\overline{k}$ an *algebraic closure* of k;

$N \subseteq G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ a *normal open subgroup* of G_k;

$\sigma \in G_k$ such that the automorphism $\iota_\sigma : N \stackrel{\sim}{\rightarrow} N$ of N given by *conjugating* by σ is *not* inner.

(One verifies immediately that, for instance, if k is a *number field* or a *mixed-characteristic local field*, then such N, σ do indeed exist.)

Write

$k_N \subseteq \overline{k}$ for the subfield of N-invariants of $\overline{k}$,
$G_{k_N} \stackrel{\text{def}}{=} N \subseteq G_k$.

Then observe that if one assumes that the **functoriality** of the *étale fundamental group* holds *even in the* **absence** *of inner automorphism indeterminacies*, then the *commutative diagram of natural morphisms of schemes*

$$\begin{array}{ccc} \mathrm{Spec}(k_N) & \stackrel{\sigma}{\longrightarrow} & \mathrm{Spec}(k_N) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k) & \end{array}$$

induces a *commutative diagram of profinite groups*

$$\begin{array}{ccc} G_{k_N} & \stackrel{\iota_\sigma}{\longrightarrow} & G_{k_N} \\ & \searrow \quad \swarrow & \\ & G_k & \end{array}$$

— which (since the natural inclusion $N = G_{k_N} \hookrightarrow G_k$ is *injective*!) implies that ι_σ is the *identity automorphism*, in *contradiction* to our assumption concerning σ!

· As a consequence of the *inter-universality* considerations discussed above (e.g., the need to work with *abstract topological groups*!), one must consider various **reconstruction algorithms** in IUT. Since reconstruction of an object is *never "set-theoretically on the nose"*, but rather always *up to (a necessarily indeterminate!) isomorphism* — whence the use of *full poly-isomorphisms*! — such reconstruction algorithms necessarily lead to **(ExtInd2) indeterminacies**. At first glance, this phenomenon may seem rather strange, but in fact, at a *purely foundational level*, this phenomenon is *completely qualitatively similar* to the indeterminacies that appear in such *classical constructions* as

· the notion of an **algebraic closure** of a field,
· **projective/inductive limits**, or
· **cohomology modules** (i.e., which arise as subquotients of *"some" indeterminate resolution*)

— cf. [EssLgc], §3.8, §3.9, §3.11.

· As a result of such **(ExtInd2) indeterminacies**, one does not obtain any *nontrivial consequences/inequalities* (cf. the "Elementary Observation" of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9) at *"stack-theoretic" intermediate steps*, i.e., even if one applies the *log-volume*!

In order to obtain *nontrivial consequences/inequalities* (cf. the "Elementary Observation" of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9), it is necessary to obtain a **"set-theoretic" closed loop**, i.e., by

· applying the **multiradial representation of the Θ-pilot**, which gives rise to the indeterminacies **(Ind1), (Ind2), (Ind3)**;

· forming the **holomorphic hull**,

· symmetrizing with respect to **vertical $\mathfrak{log}$-shifts** in the 1-column;

· and, finally, applying the **log-volume**

— as described in §6.

$$
\begin{array}{ccc}
\Pi_{\underline{v}} \twoheadrightarrow & G_{\underline{v}} & \twoheadleftarrow \Pi_{\underline{v}} \\
 & \circlearrowleft & \\
\curvearrowright & & \curvearrowright \\
 & \mathrm{Aut}(G_{\underline{v}}) & \\
\left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\mathrm{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{domain} \\ \text{of the } \Theta\text{-}\textit{link} \end{array}\right) & & \left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\mathrm{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{codomain} \\ \text{of the } \Theta\text{-}\textit{link} \end{array}\right)
\end{array}
$$

§10. Closed loops via multiradial representations and holomorphic hulls

(cf. [EssLgc], Example 2.4.6, (iii); [EssLgc], §3.10, §3.11; [ClsIUT], §2)

· We begin by observing that by *eliminating superfluous overlaps* from the *chain of* $\wedge$*'s and* $\vee$*'s* that constitutes the *essential logical structure* of IUT (cf. §6) and replacing the various *logical OR "*$\vee$*'s"* by **logical XOR "**$\dot{\vee}$**'s"**, we may think of this *essential logical str.* of IUT as consisting of a **chain of** $\wedge$**'s and** $\dot{\vee}$**'s**:

$$
\begin{array}{lcl}
A \wedge B & = & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ldots) \\
& \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ldots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ldots) \\
& \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ldots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ldots \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \ldots) \\
& & \vdots
\end{array}
$$

Recall that from the point of view of the **arithmetic** of the **field** $\mathbb{F}_2$,

$$
\begin{array}{ccc}
\wedge & \longleftrightarrow & \textbf{multiplication} \\
\dot{\vee} & \longleftrightarrow & \textbf{addition},
\end{array}
$$

while from the point of view of the **arithmetic** of the **truncated ring of Witt vectors** $\mathbb{F}_2 \times \mathbb{F}_2$ (i.e., $\mathbb{Z}/4\mathbb{Z}$),

$$
\begin{array}{ccl}
\wedge & \longleftrightarrow & \textbf{multiplication} \text{ of Teichmüller reps. of } \mathbb{F}_2 \\
(\wedge, \dot{\vee}) & \longleftrightarrow & \textbf{carry-addition} \text{ on Teichmüller reps. of } \mathbb{F}_2
\end{array}
$$

(cf. [EssLgc], Example 2.4.6, (iii)). That is to say, **carry-addition** — which may thought of as a sort of

"$\wedge$ *stacked on top of an* $\dot{\vee}$"

— is **remarkably reminiscent** of the *essential logical structure of IUT*, as well as of the fact that IUT itself is a theory concerning the explication of how the two "combinatorial dimensions" of a ring are *mutually intertwined*, i.e., how the *multiplicative structure of a ring is "stacked on top of" the additive structure of a ring*! In the case of the **chain of** $\wedge$**'s and** $\dot{\vee}$**'s** that constitutes the *essential logical structure* of IUT, we observe that:

$$\wedge \quad \longleftrightarrow \quad \begin{pmatrix} \textbf{multiplicative } \boldsymbol{\Theta}\textbf{-link}; \\ \text{data } \textbf{common} \text{ to the} \\ \textit{domain/codomain} \text{ of the} \\ \Theta\textit{-link} \end{pmatrix}$$

$$\dot{\vee} \quad \longleftrightarrow \quad \begin{pmatrix} \textbf{additive log-shells} \\ \text{arising from the } \textbf{log-link}; \\ \textit{mutually exclusive distinct} \\ \textit{possibilities} \end{pmatrix}$$

Finally, relative to the analogy between IUT and crystals, it is also of interest to observe that:

$$\wedge \quad \longleftrightarrow \quad \begin{pmatrix} \text{crystals} \\ = \textbf{“}\wedge\textbf{-crystals”} \end{pmatrix}$$

$$\dot{\vee} \quad \longleftrightarrow \quad \begin{pmatrix} \textit{mutually exclusive} \\ \text{pull-backs of the} \\ \textbf{Hodge filtration} \end{pmatrix}$$

— where we recall that it is precisely the *“intertwining between these* $\wedge$ / $\dot{\vee}$ *aspects”* that gives rise to the **Kodaira-Spencer morphism** (cf. [EssLgc], ($\wedge(\dot{\vee})$-Chn); [ClsIUT], §2).

· We conclude by reviewing once again the discussion of §6, this time taking into account the various subtleties discussed in §7, §8, §9 (cf. also [EssLgc], §3.10, §3.11).

We begin by recalling that the **log-Kummer correspondence**

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \ldots \\ & \ldots & & \searrow & \downarrow & \swarrow & & \ldots & \\ & & & & \circ & & & & \end{array}$$

— which **juggles** the **dilated** and **nondilated** underlying arithmetic dimensions of the rings involved (cf. §2)

$\mathfrak{log}$: **nondilated** unit groups $\rightleftarrows$ **dilated** value groups

— yields, by considering **invariants** with respect to the **$\mathfrak{log}$-link** and applying various **descent operations**

$$(0,0) \overset{\text{(Ind3)}}{\rightsquigarrow} (0,\circ) \overset{\text{(Ind1)}}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{\text{(Ind2)}}{\rightsquigarrow} (0,0)^{\vdash} \overset{\Theta\text{-link}}{\overset{\sim}{\rightarrow}} (1,0)^{\vdash}$$

(where we recall that the last step involving (Ind2) plays the role of **fixing** the vertical coordinate, so that (Ind1), (Ind2) are **not mixed** with (Ind3) — cf. the discussion of "$\mathbb{C}^{\times}\backslash GL_2^{+}(\mathbb{R})/\mathbb{C}^{\times}$" at the end of §5!), the **multiradial representation of the Θ-pilot**, up to the **indeterminacies (Ind1), (Ind2), (Ind3)**.

Then forming the **holomorphic hull** and symmetrizing with respect to **vertical $\mathfrak{log}$-shifts** in the 1-column

$$\begin{tikzcd}
\text{1-column} & & \text{1-column} \\
\vdots & & \vdots \\
\bullet & & \bullet \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, phantom, "\nearrow"] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, phantom, "\nearrow"] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[r, phantom, "\nearrow"] & & \bullet \arrow[u, "\mathfrak{log}"'] \\
\vdots & & \vdots
\end{tikzcd}$$

yields a **closed loop**, to which we may apply the **log-volume** to obtain **"set-theoretic" consequences/inequalities** (cf. the "Elementary Observation" of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9).

Here, we recall that the repeated introduction of **"stack-theoretic" (ExtInd2) indeterminacies**

$$\begin{array}{ccc}
\Pi_{\underline{v}} \twoheadrightarrow & G_{\underline{v}} & \twoheadleftarrow \Pi_{\underline{v}} \\
 & \circlearrowleft & \\
\curvearrowright & & \curvearrowright \\
 & \mathrm{Aut}(G_{\underline{v}}) & \\
\left(\begin{array}{c}\text{some portion of}\\ \text{the } \textit{Frobenius-like}\\ \text{local data at}\\ \underline{v} \text{ of the}\\ (\Theta^{\pm\mathrm{ell}}\text{NF-})\\ \text{Hodge theater}\\ \text{in the } \textit{domain}\\ \text{of the } \Theta\text{-}\textit{link}\end{array}\right) & & \left(\begin{array}{c}\text{some portion of}\\ \text{the } \textit{Frobenius-like}\\ \text{local data at}\\ \underline{v} \text{ of the}\\ (\Theta^{\pm\mathrm{ell}}\text{NF-})\\ \text{Hodge theater}\\ \text{in the } \textit{codomain}\\ \text{of the } \Theta\text{-}\textit{link}\end{array}\right)
\end{array}$$

— especially in the context of various *reconstruction algorithms* — allows us to achieve the *central goal* of **exhibiting** the (value group portion of the) **Θ-pilot** at $(0,0)$ (i.e., which appears in the Θ-link!) as **one of the possibilities** within a **container** arising from the **RHS** of the **Θ-link**. Moreover, the *essential logical structure*

$$\begin{aligned}
A \wedge B \quad &= \quad A \wedge (B_1 \vee B_2 \vee \dots) \\
&\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots) \\
&\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots \vee B_1'' \vee B_2'' \vee \dots) \\
&\qquad\qquad\qquad \vdots
\end{aligned}$$

underlying the **closed loop** referred to above means that there are **no** issues of **"diagram commutativity"** or **"log-vol. compatibility"** to contend with:

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & \bullet & = & = & \stackrel{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & & (& \vee & \bullet & =) & = & \stackrel{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & (& \vee & \vee & \bullet & = & =) & \stackrel{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & (& \vee & \vee & \vee & \bullet & = & = & \stackrel{\wedge}{=}) & = & = & \bullet & \\
\rightsquigarrow & & & (& \vee & \vee & \vee & \vee & \bullet & = & = & \stackrel{\wedge}{=} & =) & = & \bullet & \\
\rightsquigarrow & & (& \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \stackrel{\wedge}{=} & = & =) & \bullet & \\
\rightsquigarrow & (& \vee & \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \stackrel{\wedge}{=} & = & = & \bullet &)
\end{array}
$$

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